return to updates

A Conundrum

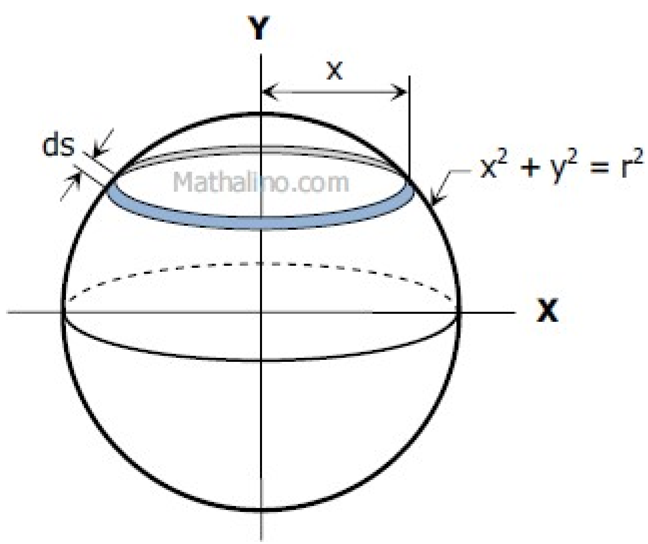
concerning the area of a sphere

by Miles Mathis

Since the surface area of a sphere and the surface area of an open cylinder of equal height are both $4\pi r^2$, let us look at the integrals for both. We can find an infinitesimal band on the surface of either which has the area

$dA = 2\pi x ds$

Where *ds* is the width of the single band and *x* is the radius at that height. In either case, we should be able to find the surface area by integrating from top to bottom, and that is in fact what is done in modern proofs. For the cylinder, *x* is a constant at all heights—equaling the radius *r*—so all we need is the total height, which in this case is 2*r*. For the equivalent sphere, *x* is not a constant, but we still integrate from bottom to top. Normally, the integration is only done on half the sphere, which is doubled, but the idea is the same.



Here is the conundrum. Since with the sphere, all values of *x* except the equatorial value are less than the radius *r,* we must sum *more* infinitesimal bands with the sphere than with the cylinder. Either that or the bands are not the same width.

To see what I mean, let us propose the opposite. Let us propose that the sphere and cylinder are composed of the same number of bands bottom to top. Since they are the same height, each *ds* will be the same. But if each *ds* is the same, and the *x*'s are different, the *dA*'s must be different.

If the dA's are different, but there are the same number of them, how can the total A's be the same? Remember

$A = \Sigma dA$
$A_S = A_C$
$\Sigma dA_S = \Sigma dA_C$
$\Sigma 2\pi x_S\, ds = \Sigma 2\pi x_C ds$?

That can't work, because all the x_C's are equal to r, but most of the x_S's are less than r.

Therefore, there must be more infinitesimal bands in the sphere than in the equivalent cylinder. Or the bands are of varying width.

Which is also a conundrum, since according to the calculus, that can't be. If you integrate from zero to $2r$, or from zero to π (or from zero to π/2, with the hemisphere), you cannot integrate one number of infinitesimals in one case and another number in another case. An infinitesimal is not a variable, one that can change value from one curve to another. Or, *ds* should equal *ds*. But here, the *ds* of the sphere does not equal the *ds* of the cylinder. Either the size of each *ds* is varying, or the number of *ds*'s is varying.

If this is in fact happening, the calculus should have a way of monitoring this variation. As it is, the variation is simply ignored. In the two proofs, the *ds*'s are labeled in the same way, so we assume they are acting the same way. It turns out they aren't, and the proofs don't tell us why. Nothing in the calculus tells us why.

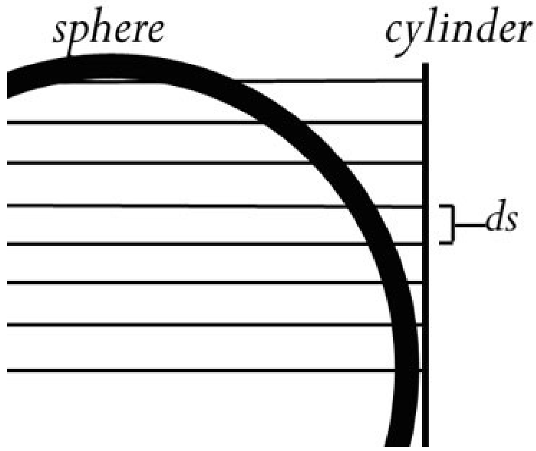
This means that the proofs are fudged. Taken separately, the proofs for area of a cylinder and area of a sphere appear to work, as we see at mathalino.com. We can only see the problem clearly if we compare the two proofs, as I just did here.

What is happening is that with the sphere, the smaller *x* is offset by the larger *ds*, but the current integral math has no way of representing that fact. It is completely invisible beneath the proofs. Yes, the value of *ds* is actually changing within the integral. It is getting larger as we go up and down on the sphere, away from the equator.

In this way, the proof of area of a sphere by integration is ignoring the differentials, just as in a planetary orbit. I have shown in previous papers that the variations in the Moon's orbit are ignored by always pointing to a sum or integral. The integration for the Moon's orbit works, so the current scientists claim there is no problem. But I have shown the problem exists in the differentials, which are varying in mysterious and unexplainable ways. Basically, if we study the differentials, we find the Moon's tangential velocity varying with no mechanical explanation. Given a gravity-only field, the Moon cannot vary its tangential velocity. But since the current proof of the orbit is an integral, this problem is hidden. The problem has a simple solution, one that requires no mysticism of any kind, but to get to this solution we have to study the differentials, not the integral.

As you see, the proof of the area of the sphere fails in the same way. By using an integral solution, the disclarities in the proof are covered over. It takes a very close analysis of the math to discover that *ds* is changing for different values of *x*.

We can see this must be the case by studying a different diagram.



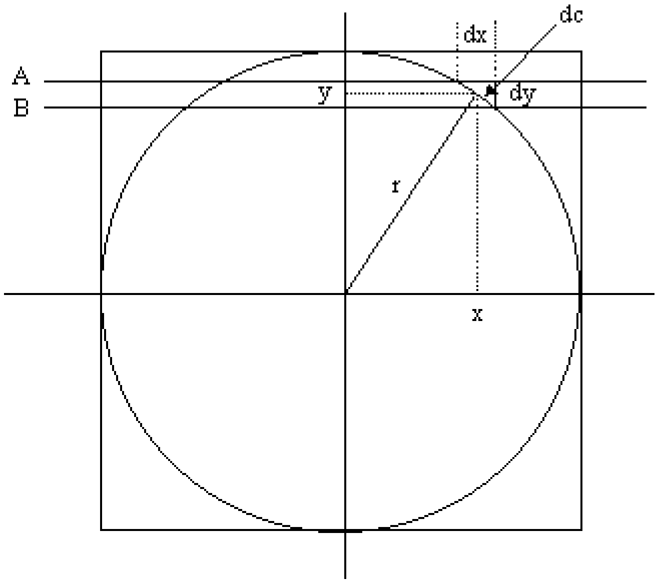
As you see, if we represent the sphere as the sum of tiny cylinders, the arc length on the sphere gets progressively longer as we go higher. The *ds*'s on the cylinder are equal, but the *ds*'s on the sphere are not. But the current and historical proofs try to ignore that. Not only do they solve by methods that sum over that variance, the methods implicitly deny the variance.

If you study the proof at mathalino.com, which is the top-listed current integral proof on the internet, the only place you get an inkling of this problem is in the equation

$$ds = \sqrt{[1 + (dy/dx)^2]}dx$$

That looks like a calculus equation, but it isn't. None of those *d*'s actually mean anything, and we aren't finding any derivatives here (yet). The trailing *dx* is confusing in that regard, because most people will want to read it as “with respect to x.” But that isn't what this equation means. This equation is just applying the Pythagorean theorem to a right triangle with *ds* as the hypotenuse. The equation isn't calculus, it is trig. That last *dx* is a real length, as in this diagram:

figure 3



The equation $ds = \sqrt{[1 + (dy/dx)^2]}dx$ is representing that little triangle, using the Pythagorean theorem, as so:

$ds^2 = dy^2 + dx^2$
$ds = \sqrt{[dy^2 + dx^2]}$
$ds = \sqrt{[1 + (dy/dx)^2]}dx$

The trailing *dx* is simply removed from within the brackets, using the laws of algebra. But rather than putting it up front, as would be more normal, this mathematician at mathalino.com lets it trail, so it looks like a calculus differential or infinitesimal.

You see, his equations and definitions aren't quite right. He is pushing them. He has a broken link to "length of arc in integral calculus," and we see why that link is broken. If you go to other integral sites, you find that the equation is normally written like this:

$ds/dx = \sqrt{[1 + (dy/dx)^2]}$

With the *dx* under the *ds*. And that initial ratio is defined as "the instantaneous rate of change of arc length *s* per unit change in *x*." So the equation written that way is now calculus, not trig. And it could also be written as

$s' = \sqrt{[1 + (dy/dx)^2]}$

Dropping the *dx* altogether.

Is s' the same as infinitesimal *s*? The calculus does not tell us. It doesn't want you to ask that question. Because if you do, you might ask why s' is the rate of change, and *ds* isn't. In this proof, *ds* has not been defined as any rate of change or ratio. It is defined as the arc length, but it is really a chord in the math and diagrams—since we have shrunk it down to straighten it out.

We see the confusion when mathalino starts integrating.

$A = 2 \int 2\pi\, x ds$
$A = 4\pi \int x\sqrt{[1 + (dy/dx)^2]}dx$

He has included the *dx*, which is fine—it is normal notation in an integral. But as notation in an integral, that *dx* doesn't allow substitution into the equation, as if the *dx* were still an infinitesimal. In integral notation, the *dx* means “I am integrating with respect to x.” It doesn't mean “a further very small x.”

Despite this, we find mathalino substituting in values for *dx*, as in

$dx = r \cos\theta\, d\theta$

Mathalino doesn't tell us where his angle θ is coming from in his diagrams, but it doesn't matter since you can't substitute anything in for *dx* when it is used in an integral like this. Since *dx* means “with respect to *x*” here, and $r \cos\theta\, d\theta$ doesn't mean “with respect to *x*,” the substitution is disallowed.

The use of *dy/dx* has also changed, because in the trig equation it means “the *y* side of the triangle over the *x* side of the triangle.” Here after the integral sign, it means “the instantaneous rate of change of *y* per unit change in *x”* A rate of change and a real length are not mathematically equivalent, so we are being fudged here. For mathalino then does this:

$x^2 + y^2 = r^2$ (from the first diagram above)
$y = \sqrt{[r^2 - x^2]}$
$dy/dx = -2x/\{2\sqrt{[r^2 - x^2]}\}$

Aha! See what he just did? He just differentiated *y* with respect to *x*. So that last *dx* **does** mean “with respect to *x*.” But he then substitutes that value into his already existing ratio *dy/dx*, where *dx* **does not** mean “with respect to *x*.” Like this:

$(dy/dx)^2 = x^2/[r^2 - x^2]$
$A = 4\pi \int x\sqrt{\{1 + [x^2/(r^2 - x^2)]\}}dx$

You see how he has just conflated the two different forms of *dy/dx?* He has done a straight substitution from one to the other, although they aren't the same. Remember, *dy/dx* was originally the sides of a real (though small) triangle. But when he found the derivative of *y*, *dy/dx* no longer meant the ratio of one real length to another. In that case, *dy/dx* means the rate of change of *y* with respect to the rate of change of *x*. Again, the substitution is disallowed.

You will say, “What in the world are you talking about? This is exactly how the calculus was invented historically. This is a form of Pascal's triangle, and the calculus was invented by Leibniz and Newton using a proof along these lines.”

Yes, and that is why all of calculus is infected by these fake proofs, no matter how old they are. The question isn't whether Leibniz or Pascal invented these proofs, or how famous those gentlemen still are. The question is whether they are right. As I am showing, they aren't. This current integral proof is fudged, and it is fudged because it is taken from older proofs that are also fudged. Mathalino didn't invent these fudges, he is just doing a copyjob on someone more famous, from whom he learned this fake proof. Calculus contains these terrible disclarities and always has.

In the equation $ds = \sqrt{[1 + (dy/dx)^2]}dx$, the only way *ds* can take different values is if *dy/dx* is taking different values. But *dy/dx* can take different values only if *dy* or *dx* are. In our current problem, *dy* is defined as constant, since it is that very small change in height of our tiny cylinders. You will say, “Then *dx* must be changing. Nothing wrong with that.” Yes, *dx* is changing at different values of x, but that is far from clear in the current proof. For one thing, *dx* is going to zero, $\Delta x \rightarrow 0$, to take us to the limit, and most people don't realize that *dx* can still vary at the limit.* You would think something at its limit would be stable, right? A varying *dx* conflicts with the whole limit idea, which is precisely why the variation is covered up on purpose. Just think about it: if *dx* is varying at the limit, then it can't really be at the limit. Or, it can be at the limit only when we are at the equator of the circle. Since *dx* is larger as we move out from the equator, it can't be at the limit in all positions. If it is at the lower limit at the equator, it must be somewhat above the limit at the poles, no?

We can see that this is true just by mathalino's diagram way above. He draws the blue band *dA* at some place on the sphere, but it doesn't matter to his solution where he draws it. It doesn't matter because in his solution, all *ds*'s seem to be the same. But as we see from my diagram, all *ds*'s are not the same. Only the vertical or *y* component of all *ds*'s is the same. But the arc length at different *ds*'s is very different. Since the area *dA* is determined by the arc length, not the vertical component, mathalino's analysis and proof must fail.

In this way, we have finally touched bottom. When we compose a sphere of infinitesimal rings like this, the rings do indeed have the same height. All the *dh*'s are the same, which allows us to stick to the rules of the calculus, where all infinitesimals equal all others. But for different values of *x*, the *dh*'s are not equivalent to the *ds*'s. The *ds*'s should be the arc lengths, and even at the limit, the arc length is not a straight function of the *dh*. No matter how small you go, the arc lengths at the top and bottom of the sphere will be larger than the arc lengths near the equator. Going to a limit does not nullify this truth.

So the current integral proof is fudged just like Pascal's historical proof (only worse). It conflates various forms of *dx*, making multiple illegal substitutions. The current proof might be called an improvement on Pascal's, since the fudge is much better hidden. It took me just a quick look to see Pascal's error; it took me a second and third look to unwind the integral proof. Or, the current proof might be called much worse, since it is pushed in multiple places to bring it back to the known answer. Pascal's proof was pushed only once.

What all this means is that Archimedes' proof by demonstration is actually superior to Pascal's proof and the current proof. Although Archimedes' proof isn't what we would call advanced, it wins by being consistent and straightforward. It isn't fudged at any point. Its primary point of superiority is that this variance of arc lengths remains clear in his proof. Because Archimedes proves the sphere by comparison to the outlying cylinder, you can see for yourself that the arc length is varying, and Archimedes never tries to hide or deny that. He never implies that we can ignore this variance as we increase the number of frusta. In fact, his very use of inscribed frusta instead of inscribed cylinders is his saving grace, since frusta by definition include this variance where cylinders do not. The frustum is part of a cone, so although the frustum has no curvature on its outer face, the length of this outer face must get larger as we go away from the equator (provided we use frusta of the same height). He uses the chord instead of the arc, that is, but since the chord is proportional to the arc, this will not affect his solution at any limit.

Archimedes proof is also superior because in it, it is clear that at each value of *x*, a greater *dx* is making up for a lesser *x*. As we go up the sphere from equator to pole, *x* gets smaller and *dx* gets larger (see *fig. 3*). In his proof, we see for ourselves that $(r)(dy) = (x)(ds)$. Pascal's proof also includes this equality, but of course he got it from Archimedes. The current integral proof hides this equality.

*This is why I have thrown out all limits, approaches to zero, and modern notation in my simplified calculus. By using a constant differential of one, and refusing to use sloppy notation, I drive around these substitution problems as well as all limit problems.